Zhao and Cheung *Journal of Inequalities and Applications* 2011, **2011**:7
http://www.journalofinequalitiesandapplications.com/content/2011/1/7

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

On some Opial-type inequalities

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Abstract

In the present paper we establish some new Opial-type inequalities involving higher-order partial derivatives. Our results in special cases yield some of the recent results on Opial's inequality and also provide new estimates on inequalities of this type.

MR (2000) Subject Classification 26D15

Keywords: Opial?'?s inequality, Opial-type integral inequalities, H?ö?lder?'?s inequality

1 Introduction

In the year 1960, Opial [1] established the following integral inequality:

Theorem 1.1. *Suppose* $f \in C^1[0, h]$ *satisfies* $f(0) = f(h) = 0$ *and* $f(x) > 0$ *for all* $x \in (0, h)$. *Then the integral inequality holds*

$$\int_0^h |f(x)f'(x)|\,\mathrm{d}x \le \frac{h}{4}\int_0^h (f'(x))^2\mathrm{d}x, \tag{1.1}$$

where this constant $\frac{h}{4}$ *is best possible.*

Opial's inequality and its generalizations, extensions and discretizations play a fundamental role in establishing the existence and uniqueness of initial and boundary value problems for ordinary and partial differential equations as well as difference equations [2-6]. The inequality (1.1) has received considerable attention, and a large number of papers dealing with new proofs, extensions, generalizations, variants and discrete analogues of Opial's inequality have appeared in the literature [7-22]. For an extensive survey on these inequalities, see [2,6]. For Opial-type integral inequalities involving high-order partial derivatives see [23-27]. The main purpose of the present paper is to establish some new Opial-type inequalities involving higher-order partial derivatives by an extension of Das's idea [28]. Our results in special cases yield some of the recent results on Opial's-type inequalities and provide some new estimates on such types of inequalities.

2 Main results

Let $n \ge 1$, $k \ge 1$. Our main results are given in the following theorems.

Theorem 2.1 *Let* $x(s, t) \in C^{(n-1)}[0, a] \times C^{(k-1)}[0, b]$ *be such that* $\frac{\partial^i}{\partial\sigma^i}x(0,\tau) = 0$, $\frac{\partial^j}{\partial\tau^j}x(\sigma,0) = 0$, $\sigma \in [0, s]$, $\tau \in [0, t]$, $0 \le i \le n-1$, $0 \le j \le k-1$. *Further, let* $\frac{\partial^{n-1}}{\partial s^{n-1}}x(s,t)$, $\frac{\partial^{k-1}}{\partial t^{k-1}}x(s,t)$ *be absolutely continuous, and* $\int_0^a\int_0^b |x^{(n,k)}(s,t)|^2\mathrm{d}s\,\mathrm{d}t < \infty$. *Then*

$$\int_0^a \int_0^b \left| x(s,t) \cdot x^{(n,k)}(s,t) \right| \mathrm{d}s\,\mathrm{d}t \le c_{n,k} \cdot a^n b^k \cdot \int_0^a \int_0^b \left| x^{(n,k)}(s,t) \right|^2 \mathrm{d}s\,\mathrm{d}t, \tag{2.1}$$

where

$$x^{(n,k)}(s,t) = \frac{\partial^n}{\partial s^n}\left(\frac{\partial^k}{\partial t^k} x(s,t)\right),$$

and

$$c_{n,k} = \frac{1}{4n!k!}\left(\frac{2nk}{(2n-1)(2k-1)}\right)^{\frac{1}{2}}.$$

Proof. For σ integration by parts (n - 1)-times and in view of $\frac{\partial^i}{\partial \sigma^i} x(0,\tau) = 0$, $\frac{\partial^j}{\partial \tau^j} x(\sigma, 0) = 0$, $0 \le i \le n - 1$, $0 \le j \le k - 1$ we have

$$\begin{aligned} x(s,t) &= \frac{(-1)^n}{(n-1)!}\int_s^0 (\sigma - s)^{n-1} \frac{\partial^n}{\partial \sigma^n} x(\sigma, t)\,\mathrm{d}\sigma \\ &= \frac{(-1)^{2n-1}}{(n-1)!}\int_s^0 (s-\sigma)^{n-1} \frac{\partial^n}{\partial \sigma^n} x(\sigma,t)\,\mathrm{d}\sigma = \frac{1}{(n-1)!}\int_0^s (s-\sigma)^{n-1}\frac{\partial^n}{\partial\sigma^n} x(\sigma,t)\,\mathrm{d}\sigma \\ &= \frac{1}{(n-1)!(k-1)!}\int_0^s (s-\sigma)^{n-1}\frac{\partial^n}{\partial \sigma^n}\left(\int_0^t (t-\tau)^{k-1}\frac{\partial^k}{\partial \tau^k} x(\sigma,\tau) d\tau\right)\mathrm{d}\sigma \\ &= \frac{1}{(n-1)!(k-1)!}\int_0^s\int_0^t (s-\sigma)^{n-1}(t-\tau)^{k-1} \cdot x^{(n,k)}(\sigma,\tau)\,\mathrm{d}\sigma\,\mathrm{d}\tau. \end{aligned} \tag{2.2}$$

Multiplying both sides of (2.2) by $x^{(n,k)}(s, t)$ and using the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \left| x(s,t) \cdot x^{(n,k)}(s,t)\right| &\le \frac{\left|x^{(n,k)}(s,t)\right|}{(n-1)!(k-1)!}\left(\int_0^s\int_0^t (s-\sigma)^{2n-2}(t-\tau)^{2k-2}\,\mathrm{d}\sigma\,\mathrm{d}\tau\right)^{\frac{1}{2}} \\ &\quad \times \left(\int_0^s\int_0^t \left|x^{(n,k)}(\sigma,\tau)\right|^2 \mathrm{d}\sigma\,\mathrm{d}\tau\right)^{\frac{1}{2}} \\ &= \frac{1}{(n-1)!(k-1)!\sqrt{(2n-1)(2k-1)}} \cdot s^{n-\frac{1}{2}} t^{k-\frac{1}{2}} \left|x^{(n,k)}(s,t)\right| \left(\int_0^s\int_0^t \left|x^{(n,k)}(\sigma,\tau)\right|^2 \mathrm{d}\sigma\,\mathrm{d}\tau\right)^{\frac{1}{2}}. \end{aligned} \tag{2.3}$$

Thus, integrating both sides of (2.3) over t from 0 to b first and then integrating the resulting inequality over s from 0 to a and applying the Cauchy-Schwarz inequality again, we obtain

$$\begin{aligned} &\int_0^a\int_0^b \left|x(s,t)\cdot x^{(n,k)}(s,t)\right| \mathrm{d}s\,\mathrm{d}t \\ &\le \frac{1}{(n-1)!(k-1)!\sqrt{(2n-1)(2k-1)}}\left(\int_0^a\int_0^b s^{2n-1}t^{2k-1}\mathrm{d}s\,\mathrm{d}t\right)^{\frac{1}{2}} \\ &\times\left(\int_0^a\int_0^b \left|x^{(n,k)}(s,t)\right|^2\left(\int_0^s\int_0^t\left|x^{(n,k)}(\sigma,\tau)\right|^2\mathrm{d}\sigma\,\mathrm{d}\tau\right)\mathrm{d}s\,\mathrm{d}t\right)^{\frac{1}{2}} \\ &= \frac{1}{2n!}\left(\frac{2n}{2n-1}\right)^{\frac{1}{2}}\frac{1}{2k!}\left(\frac{2k}{2k-1}\right)^{\frac{1}{2}} a^n b^k \\ &\times\left(\frac{1}{2}\int_0^a\int_0^b \frac{\partial^2}{\partial s \partial t}\left\{\left(\int_0^s\int_0^t \left|x^{(n,k)}(\sigma,\tau)\right|^2 \mathrm{d}\sigma\,\mathrm{d}\tau\right)^2\right\}\mathrm{d}s\,\mathrm{d}t\right)^{\frac{1}{2}} \\ &= c_{n,k}a^nb^k\int_0^a\int_0^b\left|x^{(n,k)}(s,t)\right|^2\mathrm{d}s\,\mathrm{d}t. \end{aligned}$$

This completes the proof.

Remark 2.1. Let $x(s, t)$ reduce to $s(t)$ and with suitable modifications, Then (2.1) becomes the following inequality:

$$\int_0^a |x(t)x^{(n)}(t)|\mathrm{d}t \leq \frac{1}{2n!} \cdot \left(\frac{n}{2n-1}\right)^{\frac{1}{2}} a^n \int_0^a |x^{(n)}(t)|^2 \mathrm{d}t. \tag{2.4}$$

This is just an inequality established by Das [28]. Obviously, for $n \geq 2$, (2.4) is sharper than the following inequality established by Willett [29].

$$\int_0^a |x(t)x^n(t)|\mathrm{d}t \leq \frac{1}{2} a^n \int_0^a |x^n(t)|^2 \mathrm{d}t. \tag{2.5}$$

Remark 2.2. Taking for $n = k = 1$ in (2.1), (2.1) reduces to

$$\int_0^a \int_0^b \left| x(s,t) \cdot \frac{\partial^2}{\partial s \partial t} x(s,t) \right| \mathrm{d}s\,\mathrm{d}t \leq \frac{\sqrt{2}}{4} ab \int_0^a \int_0^b \left| \frac{\partial^2}{\partial s \partial t} x(s,t) \right|^2 \mathrm{d}s\,\mathrm{d}t. \tag{2.6}$$

Let $x(s, t)$ reduce to $s(t)$ and with suitable modifications. Then (2.6) becomes the following inequality: If $x(t)$ is absolutely continuous in $[0, a]$ and $x(0) = 0$, then

$$\int_0^a |x(t)x'(t)|\mathrm{d}t \leq \frac{a}{2} \int_0^a |x'(t)|^2 \mathrm{d}t.$$

This is just an inequality established by Beesack [30].

Remark 2.3. Let $0 \leq \alpha, \beta < n$, but fixed, and let $g(s, t) \in C^{(n-\alpha-1)}[0, a] \times C^{(k-\beta-1)}[0, b]$ be such that $\frac{\partial^i}{\partial s^i} g(0,t) = \frac{\partial^i}{\partial t^i} g(s,0) = 0$, $0 \leq i \leq n - \alpha - 1$, $0 \leq i \leq k - \beta - 1$ and suppose that $\frac{\partial^{n-\alpha-1}}{\partial s^{n-\alpha-1}} g(s,t)$, $\frac{\partial^{k-\beta-1}}{\partial t^{k-\beta-1}} g(s,t)$ are absolutely continuous, and $\int_0^a \int_0^b \left| x^{(n-\alpha,k-\beta)}(s,t) \right|^2 \mathrm{d}s\,\mathrm{d}t < \infty$.

Then from (2.1) it follows that

$$\int_0^a \int_0^b \left| g(s,t) \cdot g^{(n-\alpha,k-\beta)}(s,t) \right| \mathrm{d}s\,\mathrm{d}t \leq c_{n-\alpha,k-\beta} a^{n-\alpha} b^{k-\beta} \int_0^a \int_0^b \left| g^{(n-\alpha,k-\beta)}(s,t) \right|^2 \mathrm{d}s\,\mathrm{d}t.$$

Thus, for $g(s, t) = x^{(\alpha, \beta)}(s, t)$, where $x(s, t) \in C^{(n-1)}[0, a] \times C^{(k-1)}[0, b]$, $\frac{\partial^i}{\partial s^i} x(0,t) = 0$, $\frac{\partial^j}{\partial t^j} x(s,0) = 0$, $\alpha \leq i \leq n - 1$, $\beta \leq j \leq k - 1$, and $x^{(n-1,\, k-1)}(s, t)$ are absolutely continuous, and $\int_0^a \int_0^b \left| x^{(n,k)}(s,t) \right|^2 \mathrm{d}s\,\mathrm{d}t < \infty$, then

$$\int_0^a \int_0^b \left| x^{(\alpha,\beta)}(s,t) \cdot x^{(n,k)}(s,t) \right| \mathrm{d}s\,\mathrm{d}t \leq c_{n-\alpha,k-\beta} a^{n-\alpha} b^{k-\beta} \int_0^a \int_0^b \left| x^{(n,k)}(s,t) \right|^2 \mathrm{d}s\,\mathrm{d}t. \tag{2.7}$$

Obviously, a special case of (2.7) is the following inequality:

$$\int_0^a \int_0^b \left| x^{(k,k)}(s,t) \cdot x^{(n,n)}(s,t) \right| \mathrm{d}s\,\mathrm{d}t \leq c_{n-k,n-k} (ab)^{n-k} \int_0^a \int_0^b \left| x^{(n,n)}(s,t) \right|^2 \mathrm{d}s\,\mathrm{d}t \tag{2.8}$$

Let $x(s, t)$ reduce to $s(t)$ and with suitable modifications. Then (2.8) becomes the following inequality:

$$\int_0^a |x^{(k)}(t)x^{(n)}(t)|\mathrm{d}t \leq \frac{1}{2(n-k)!} \cdot \left(\frac{n-k}{2(n-k)-1}\right)^{\frac{1}{2}} a^{n-k} \int_0^a |x^{(n)}(t)|^2 \mathrm{d}t.$$

This is just an inequality established by Agarwal and Thandapani [31].

Theorem 2.2. *Let l and m be positive numbers satisfying $l + m > 1$. Further,* let $x(s, t) \in C^{(n-1)}[0, a] \times C^{(k-1)}[0, b]$ *be such that* $\frac{\partial^i}{\partial\sigma^i}x(0,\tau) = 0$, $\frac{\partial^j}{\partial\tau^j}x(\sigma,0) = 0$, $\sigma \in [0, s]$, $\tau \in [0, t]$, $0 \leq i \leq n - 1$, $0 \leq j \leq k - 1$ *and assume that* $\frac{\partial^{n-1}}{\partial s^{n-1}}x(s,t)$, $\frac{\partial^{k-1}}{\partial t^{k-1}}x(s,t)$ *are absolutely continuous, and* $\int_0^a \int_0^b \left|x^{(n,k)}(s,t)\right|^{l+m} \mathrm{d}s\,\mathrm{d}t < \infty$. *Then*

$$\int_0^a \int_0^b |x(s,t)|^l \left|x^{(n,k)}(s,t)\right|^m \mathrm{d}s\,\mathrm{d}t \leq c^*_{n,k} a^{nl} b^{kl} \int_0^a \int_0^b \left|x^{(n,k)}(s,t)\right|^{l+m} \mathrm{d}s\,\mathrm{d}t, \tag{2.9}$$

where

$$c^*_{n,k} = \xi^{l\xi+1} m^{\xi m} \left(\frac{kn(1-\xi)^2}{(n-\xi)(k-\xi)}\right)^{l(1-\xi)} \cdot (n!k!)^{-l}, \quad \xi = \frac{1}{l+m}.$$

Proof. From (2.2), we have

$$|x(s,t)| \leq \frac{1}{(n-1)!(k-1)!} \int_0^s \int_0^t (s-\sigma)^{n-1}(t-\tau)^{k-1} \left|x^{(n,k)}(\sigma,\tau)\right| \mathrm{d}\sigma\,\mathrm{d}\tau,$$

by Hölder's inequality with indices $l + m$ and $\frac{l+m}{l+m-1}$, it follows that

$$\begin{aligned} |x(s,t)| &\leq \frac{1}{(n-1)!(k-1)!} \left(\int_0^s \int_0^t [(s-\sigma)^{n-1}(t-\tau)^{k-1}]^{\frac{l+m}{l+m-1}} \mathrm{d}\sigma\,\mathrm{d}\tau\right)^{\frac{l+m-1}{l+m}} \\ &\quad \times \left(\int_0^s \int_0^t \left|x^{(n,k)}(\sigma,\tau)\right|^{l+m} \mathrm{d}\sigma\,\mathrm{d}\tau\right)^{\frac{1}{l+m}} \\ &= A s^{n-\xi} t^{k-\xi} \left(\int_0^s \int_0^t \left|x^{(n,k)}(\sigma,\tau)\right|^{l+m} \mathrm{d}\sigma\,\mathrm{d}\tau\right)^{\xi}, \end{aligned}$$

where

$$A = \left(\frac{(1-\xi)^2}{(n-\xi)(k-\xi)}\right)^{1-\xi} \frac{1}{(n-1)!(k-1)!}.$$

Multiplying the both sides of above inequality by $|x^{(n,k)}(s, t)|^m$ and integrating both sides over t from 0 to b first and then integrating the resulting inequality over s from 0 to a, we obtain

$$\begin{aligned} &\int_0^a \int_0^b |x(s,t)|^l \left|x^{(n,k)}(s,t)\right|^m \mathrm{d}s\,\mathrm{d}t \\ &\leq A^l \int_0^a \int_0^b s^{l(n-\xi)} t^{l(k-\xi)} \left|x^{(n,k)}(s,t)\right|^m \left(\int_0^s \int_0^t \left|x^{(n,k)}(\sigma,\tau)\right|^{l+m} \mathrm{d}\sigma\,\mathrm{d}\tau\right)^{l\xi} \mathrm{d}s\,\mathrm{d}t. \end{aligned}$$

Now, applying Hölder's inequality with indices $\frac{l+m}{l}$ and $\frac{l+m}{m}$ to the integral on the right-side, we obtain

$$
\begin{aligned}
\int_0^a \int_0^b |x(s,t)|^l \left|x^{(n,k)}(s,t)\right|^m \mathrm{d}s\,\mathrm{d}t &\le A^l \left(\int_0^a \int_0^b s^{(n-\xi)(l+m)} t^{(k-\xi)(l+m)}\,\mathrm{d}s\,\mathrm{d}t \right)^{\frac{l}{l+m}} \\
&\times \left(\int_0^a \int_0^b \left|x^{(n,k)}(s,t)\right|^{m+l} \left(\int_0^s \int_0^t \left|x^{(n,k)}(\sigma,\tau)\right|^{l+m} \mathrm{d}\sigma\,\mathrm{d}\tau \right)^{\frac{l}{m}} \mathrm{d}s\,\mathrm{d}t \right)^{\frac{m}{l+m}} \\
&= A^l \left(\int_0^a \int_0^b s^{(n-\xi)(l+m)} t^{(k-\xi)(l+m)}\,\mathrm{d}s\,\mathrm{d}t \right)^{\frac{l}{l+m}} \\
&\times \left(\frac{m}{l+m} \int_0^a \int_0^b \frac{\partial^2}{\partial s \partial t} \left(\int_0^s \int_0^t \left|x^{(n,k)}(\sigma,\tau)\right|^{l+m} \mathrm{d}\sigma\,\mathrm{d}\tau \right)^{\frac{l}{m}+1} \mathrm{d}s\,\mathrm{d}t \right)^{\frac{m}{l+m}} \\
&= A^l \left(\frac{\xi^2}{kn} \right)^{\xi l} (m\xi)^{m\xi} a^{nl} b^{kl} \int_0^a \int_0^b \left|x^{(n,k)}(s,t)\right|^{l+m} \mathrm{d}s\,\mathrm{d}t \\
&= c^*_{n,k} a^{nl} b^{kl} \int_0^a \int_0^b \left|x^{(n,k)}(s,t)\right|^{l+m} \mathrm{d}s\,\mathrm{d}t.
\end{aligned}
$$

This completes the proof.

Remark 2.4. Let $x(s, t)$ reduce to $s(t)$ and with suitable modifications. Then (2.9) becomes the following inequality:

$$
\int_0^a \left|x(t)\right|^l \left|x^{(n)}(t)\right|^m \mathrm{d}t \le \xi m^{m\xi} \left(\frac{n(1-\xi)}{n-\xi} \right)^{l(1-\xi)} (n!)^{-l} a^{nl} \int_0^a \left|x^{(n)}(t)\right|^{l+m} \mathrm{d}t. \quad (2.10)
$$

This is an inequality given by Das [28]. Taking for $n = 1$ in (2.10), we have

$$
\int_0^a |x(t)|^l |x'(t)|^m \mathrm{d}t \le \frac{m^{m/(l+m)}}{l+m} a^l \int_0^a |x'(t)|^{m+l} \mathrm{d}t. \quad (2.11)
$$

For $m, l \ge 1$ Yang [32] established the following inequality:

$$
\int_0^a |x(t)|^l |x'(t)|^m \mathrm{d}t \le \frac{m}{l+m} a^l \int_0^a |x'(t)|^{m+l} \mathrm{d}t. \quad (2.12)
$$

Obviously, for $m, l \ge 1$, (2.11) is sharper than (2.12).

Remark 2.5. For $n = k = 1$; (2.9) reduces to

$$
\int_0^a \int_0^b |x(s,t)|^l \left| \frac{\partial^2}{\partial s \partial t} x(s,t) \right|^m \mathrm{d}s\ \mathrm{d}t \le c^*_{1,1} (ab)^l \int_0^a \int_0^b \left| \frac{\partial^2}{\partial s \partial t} x(s,t) \right|^{m+l} \mathrm{d}s\,\mathrm{d}t.
$$

Let $x(s, t)$ reduce to $s(t)$ and with suitable modifications. Then above inequality becomes the following inequality:

$$
\int_0^a |x(t)|^l |x'(t)|^m \mathrm{d}t \le \xi m^{m\xi} a^l \int_0^a |x'(t)|^{m+l} \mathrm{d}t, \quad \xi = (l+m)^{-1}.
$$

This is just an inequality established by Yang [32].

Remark 2.6. Following Remark 2.3, for $x(s, t) \in C^{(n-1)}[0, a] \times C^{(k-1)}[0, b]$, $\frac{\partial^j}{\partial t^j}x(s,0) = 0$, $\frac{\partial^j}{\partial t^j}x(s,0) = 0$, $\alpha \le i \le n-1$, $\beta \le j \le k-1$ and $x^{(n-1,k-1)}(s, t)$ are absolutely continuous, and $\int_0^a \int_0^b \left|x^{(n,k)}(s,t)\right|^{l+m} \mathrm{d}s\,\mathrm{d}t < \infty$, it is easy to obtain that

$$\int_0^a \int_0^b \left|x^{(\alpha,\beta)}(s,t)\right|^l \cdot \left|x^{(n,k)}(s,t)\right|^m \mathrm{d}s\,\mathrm{d}t \le c^*_{n-\alpha,k-\beta} a^{l(n-\alpha)} b^{l(k-\beta)} \int_0^a \int_0^b \left|x^{(n,k)}(s,t)\right|^{l+m} \mathrm{d}s\mathrm{d}t \quad (2.13)$$

Obviously, a special case of (2.14) is the following inequality:

$$\int_0^a \int_0^b \left|x^{(k,k)}(s,t)\right|^l \cdot \left|x^{(n,n)}(s,t)\right|^m \mathrm{d}s\,\mathrm{d}t \le c^*_{n-k,n-k} (ab)^{l(n-k)} \int_0^a \int_0^b \left|x^{(n,n)}(s,t)\right|^{l+m} \mathrm{d}s\,\mathrm{d}t \quad (2.14)$$

Let $x(s, t)$ reduce to $s(t)$ and with suitable modifications, then (2.14) becomes the following inequality:

$$\begin{aligned}
&\int_0^a \left|x^{(k)}(t)\right|^l \left|x^{(n)}(t)\right|^m \mathrm{d}t \\
&\le \xi m^{m\xi} \left(\frac{(n-k)(1-\xi)}{n-k-\xi}\right)^{l(1-\xi)} ((n-k)!)^{-l} a^{(n-k)l} \int_0^a \left|x^{(n)}(t)\right|^{l+m} \mathrm{d}t, \quad \xi = (l+m)^{-1}.
\end{aligned}$$

This is just an inequality established by Agarwal and Thandapani [31].

Theorem 2.3. *Let l and m be positive numbers satisfying $l + m = 1$. Further,* let $x(s, t) \in C^{(n-1)}[0, a] \times C^{(k-1)}[0, b]$ *be such that* $\frac{\partial^i}{\partial \sigma^i}x(0,\tau) = 0$, $\frac{\partial^j}{\partial \tau^j}x(\sigma,0) = 0$, $\sigma \in [0, s]$, $\tau \in [0, t]$, $0 \le i \le n-1$, $0 \le j \le k-1$ *and assume that* $\frac{\partial^{n-1}}{\partial s^{n-1}}x(s,t)$, $\frac{\partial^{k-1}}{\partial t^{k-1}}x(s,t)$ *are absolutely continuous, and* $\int_0^a \int_0^b \left|x^{(n,k)}(s,t)\right| \mathrm{d}s\,\mathrm{d}t < \infty$. *Then*

$$\int_0^a \int_0^b |x(s,t)|^l \left|x^{(n,k)}(s,t)\right|^m \mathrm{d}s\,\mathrm{d}t \le \frac{m^m}{(n!k!)^l} a^{nl} b^{kl} \int_0^a \int_0^b \left|x^{(n,k)}(s,t)\right| \mathrm{d}s\,\mathrm{d}t. \quad (2.15)$$

Proof. It is clear that

$$\begin{aligned}
|x(s,t)| &\le \frac{1}{(n-1)!(k-1)!} \int_0^s \int_0^t (s-\sigma)^{n-1} (t-\tau)^{k-1} \left|x^{(n,k)}(\sigma,\tau)\right| \mathrm{d}\sigma\,\mathrm{d}\tau \\
&\le \frac{1}{(n-1)!(k-1)!} s^{n-1} t^{k-1} \int_0^s \int_0^t \left|x^{(n,k)}(\sigma,\tau)\right| \mathrm{d}\sigma\,\mathrm{d}\tau
\end{aligned}$$

and hence

$$\begin{aligned}
&\int_0^a \int_0^b |x(s,t)|^l \left|x^{(n,k)}(s,t)\right|^m \mathrm{d}s\,\mathrm{d}t \\
&\le \frac{1}{[(n-1)!(k-1)!]^l} \int_0^a \int_0^b s^{(n-1)l} t^{(k-1)l} \left|x^{(n,k)}(s,t)\right|^m \left(\int_0^s \int_0^t \left|x^{(n,k)}(s,t)\right| \mathrm{d}s\,\mathrm{d}t\right)^l \mathrm{d}s\,\mathrm{d}t.
\end{aligned}$$

Now applying Hölder inequality with indices $\frac{1}{l}$ and $\frac{1}{m}$, we obtain

$$\begin{aligned}\int_0^a\int_0^b |x(s,t)|^l\left|x^{(n,k)}(s,t)\right|^m \mathrm{d}s\,\mathrm{d}t &\le \frac{1}{[(n-1)!(k-1)!]^l}\left(\int_0^a\int_0^b s^{n-1}t^{k-1}\mathrm{d}s\,\mathrm{d}t\right)^l\\ &\times\left(\int_0^a\int_0^b \left|x^{(n,k)}(s,t)\right|\left(\int_0^s\int_0^t \left|x^{(n,k)}(\sigma,\tau)\right| d\sigma d\tau\right)^{\frac{l}{m}}\mathrm{d}s\,\mathrm{d}t\right)^m\\ &= \frac{1}{[(n-1)!(k-1)!]^l}\left(\frac{1}{n!k!}\right)^l a^{nl}b^{kl}\\ &\times\left(m\int_0^a\int_0^b \frac{\partial^2}{\partial s\partial t}\left\{\left(\int_0^s\int_0^t \left|x^{(n,k)}(\sigma,\tau)\right| d\sigma d\tau\right)^{\frac{l}{m}+1}\right\}\mathrm{d}s\,\mathrm{d}t\right)^m\\ &= \frac{m^m}{(n!k!)^l}a^{nl}b^{kl}\int_0^a\int_0^b \left|x^{(n,k)}(s,t)\right|\mathrm{d}s\,\mathrm{d}t.\end{aligned}$$

This completes the proof.

Remark 2.7. Let $x(s, t)$ reduce to $s(t)$ and with suitable modifications. Then (2.16) becomes the following inequality:

$$\int_0^a |x^{(k)}(t)|^l|x^{(n)}(t)|^m \mathrm{d}t \le \frac{m^m}{(n!)^l}a^{nl}\int_0^a |x^n(t)|\mathrm{d}t.$$

This is an inequality given by Das [28].

Remark 2.8. Following Remark 2.3, for $x(s, t) \in C^{(n-1)}[0, a] \times C^{(k-1)}[0, b]$, $\frac{\partial^j}{\partial t^j}x(s,0) = 0$, $\frac{\partial^j}{\partial t^j}x(s,0) = 0$, $\alpha \le i \le n-1$, $\beta \le j \le k-1$, and $x^{(n-1,k-1)}(s, t)$ are absolutely continuous, and $\int_0^a\int_0^b \left|x^{(n,k)}(s,t)\right|\mathrm{d}s\,\mathrm{d}t < \infty$, from (2.16), it is easy to obtain that

$$\begin{aligned}&\int_0^a\int_0^b \left|x^{(\alpha,\beta)}(s,t)\right|^l\cdot\left|x^{(n,k)}(s,t)\right|^m \mathrm{d}s\,\mathrm{d}t\\ &\le \frac{m^m}{[(n-\alpha)!(k-\beta)!]^l}a^{l(n-\alpha)}b^{l(k-\beta)}\int_0^a\int_0^b \left|x^{(n,k)}(s,t)\right|\mathrm{d}s\,\mathrm{d}t.\end{aligned}\tag{2.16}$$

Let $x(s, t)$ reduce to $s(t)$ and with suitable modifications, then (2.16) becomes the following inequality:

$$\int_0^a |x^{(k)}(t)|^l|x^{(n)}(t)|^m \mathrm{d}t \le \frac{m^m}{((n-k)!)^l}a^{(n-k)l}\int_0^a |x^n(t)|\mathrm{d}t,\ l+m=1.$$

This is an inequality given by Das [28].

Acknowledgements

The authors express their grateful thanks to the referee for his many very valuable suggestions and comments. Research of Chang-Jian Zhao was supported by National Natural Science Foundation of China (10971205). Research of Wing-Sum Cheung was partially supported by a HKU URC grant.

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C-JZ and W-SC jointly contributed to the main results Theorems 2.1, 2.2, and 2.3. Both authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

Received: 28 December 2010 Accepted: 17 June 2011 Published: 17 June 2011

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doi:10.1186/1029-242X-2011-7
Cite this article as: Zhao and Cheung: **On some Opial-type inequalities.** *Journal of Inequalities and Applications* 2011 **2011**:7.